सत्यं शिवमानम्

WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 2nd Semester Examination, 2023

MTMACOR03T-MATHEMATICS (CC3)

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: $2\times5=10$

(a) State Supremum property and Archimedean property of R, the set of all real numbers. 1+1

(b) Is the set $\{x\in R:\ \sin x\neq 0\}$ open in R? Justify your answer.

(c) Verify Bolzano-Weierstrass theorem for the set $\left\{\frac{n}{n+1}:\ n\in\mathbb{N}\right\}$.

(d) Prove that the sequence $\{x_n\}$ where $x_n=\frac{1}{1\cdot3}+\frac{1}{3\cdot5}+\cdots\cdots+\frac{1}{(2n-1)(2n+1)}$ is bounded.

(e) If $A=[-1,\ 4)$ and $B=(2,\ 5]$, is $A\cup B$ compact? Give reasons.

(f) Show that the sequence $\{x_n\}$ is a null sequence where $x_n=\frac{n!}{n^n}$.

(g) Use comparison test to examine the convergence of the series:

$$\frac{1}{1\cdot2^2}+\frac{1}{2\cdot3^2}+\frac{1}{3\cdot4^2}+\cdots\cdots$$

(h) Test the convergence of the series:

$$1-\frac{2^2}{2!}+\frac{3^3}{3!}-\frac{4^4}{4!}+\cdots\cdots$$

2. (a) Let A and B be two non-empty bounded sets of real numbers. Let $C=\{x+y\ :\ x\in A,\ y\in B\}$. Show that $\sup C=\sup A+\sup B$. 3

(b) If S be a subset of R, then prove that interior of S is an open set. 2

(c) Prove that the set $\mathbb{Q}$ of rational numbers is enumerable. 3

3. (a) If $S=\{(-1)^m+\frac{1}{n};\ m\in N,\ n\in N\}$, then find the derived set of S. Is S a closed set? Justify your answer. 3

(b) If G is an open set in R then prove that $R-G$ is closed. 2

(c) Let $S=\bigcup_{n=1}^{\infty}I_n$, where $I_n=\left\{x\in R:\frac{1}{2^n}\leq x\leq 1\right\}$. Is the set S closed? Justify your answer. 3

4. (a) Prove that every compact subset of R is closed and bounded. 5
(b) Give an example of a set which is closed, but not compact. Give reasons. 1
(c) Prove that the intersection of two compact sets in R is compact. 2

5. (a) State and prove Sandwich theorem for convergence of a sequence and use it to prove that $\lim_{n\to\infty}(2^n+3^n)^{1/n}=3$. 1+2+2
(b) If $u_1>0$ and $u_{n+1}=\frac{1}{2}\left(u_n+\frac{9}{u_n}\right)$, $\forall\, n\geq 1$, then show that $\{u_n\}$ is monotonically decreasing and bounded below. Is it convergent? 3

6. (a) If a sequence $\{u_n\}$ converges to l, then prove that every subsequence of $\{u_n\}$ converges to l. 2
(b) If the n-th term of the sequence $\{u_n\}$ is given by $u_n=(-1)^n+\sin\frac{n\pi}{4}$, $n=1, 2, 3, \cdots\cdots$, then find two subsequences of $\{u_n\}$, one converging to the upper limit and the other converging to the lower limit. Is the sequence convergent? Give reasons. 3
(c) Show that $\lim_{n\to\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdots(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdots 2n}=0$. 3

7. (a) Prove that every convergent sequence is bounded. Is the converse true? Give reasons. 2+1
(b) Using definition of Cauchy sequence, show that the sequence $\{\frac{1}{n}\}$ is a Cauchy sequence. 2
(c) Prove or disprove: A monotone sequence of real numbers having a convergent subsequence is convergent. 3

8. (a) State and prove Leibnitz test for convergence of an alternating series. 1+3
(b) Use this to test the convergence of the series $\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n\log n}$. 2
(c) Define conditionally convergent series with example. 1+1

9. (a) Use Cauchy's integral test to show that $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges for $p>1$ and diverges for $p\leq 1$. 3
(b) Test the convergence of the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}.(n!)^2.7^n}{(2n)!}$. 3
(c) If $\sum_{n=1}^{\infty}a_n$ is a convergent series of positive real numbers, will the series $\sum_{n=1}^{\infty}a_{2n}$ be convergent? Justify your answer. 2

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